Introduction

During our normal lessons, we frequently come across triangles with integral sides and sometimes with integral areas. It is obvious that a triangle with integral sides may not always have an integral area and vice versa. This observation prompted us to think about an interesting problem – is there a way to find all possible triangles with integral sides, together with an integral area?

This article is based on the entry submitted by Anglo-Chinese School (Independent) in the 1995 Secondary Science Fair held in conjunction with the 18th Singapore Youth Science Fortnight from 25 to 27 May 1995. The authors are upper secondary students of ACS(I). Their project "Heronian Triangles" won the 2nd prize in the Mathematics Section.

Overview

Heron of Alexandria, a Greek mathematician and scientist, derived the well-known formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where A is the area and $a, b, c,$ are the sides of the triangle. Here, s is the semi-perimeter of the triangle, i.e. $s = (a + b + c)/2$. Using this formula, the area of a triangle can be easily computed, especially when the sides of the triangle, a, b, c, are all integers.

We will call a triangle a ***Heronian triangle*** if its three sides $a, b, c,$ and its area A are all integers. The triple (a, b, c) is then called a ***Heronian triple***.

For example, a triangle with integral sides 4, 13, 15 is a Heronian triangle, and the triple (4, 13, 15) is a Heronian triple.

Heron's formula does not seem to work very well if we want to find *all* possible Heronian triangles. We have to resort to other means in order to determine all possible Heronian triangles.

TRIANGLES

by Herman Chow, Tang Chi Ho Eric and Fung Chi Yeung

Objectives

Stated explicitly, the objective of our project is ***to find all possible Heronian triangles***. We will attempt to find a general formula that generates all Heronian triples. In the process of investigation, we hope to discover some properties which are related to the Heronian triangles. A related question that comes to mind: Are there Heronian triangles which are equilateral, right-angled or isosceles?

In order to facilitate the investigation, each of the different cases will be studied separately.

Methods & Findings

Case 1 Equilateral triangles

Theorem ***There is no equilateral triangle that is Heronian.***

The area of a triangle can be given by the formula

$$A = (1/2)ab\sin C.$$

For an equilateral triangle $C = 60^{\circ}$, $\sin C = \sqrt{3}/2$.
Since $\sqrt{3}$ is an irrational number and a and b are integers, the area A is also an irrational number. Thus A is not an integer. This proves that it is impossible to have a Heronian triangle which is equilateral.

Case 2 Right-angled triangles

Theorem ***All Pythagorean triangles are right-angled Heronian triangles.***

The Pythagoras' theorem states the most famous property of right-angled triangles: $a^2 + b^2 = c^2$. We define a ***Pythagorean triangle*** as a right-angled triangle whose sides are integers. Then the triple (a, b, c) is called a ***Pythagorean triple***. For example (3, 4, 5) is the simplest Pythagorean triple.

By definition a right-angled Heronian triangle is a Pythagorean triangle. In order to prove the above theorem, we only have to prove that one of the non-hypothenus sides of a Pythagorean triangle must be even, so that its area will be integral.

Proof: Assume that both a and b are odd.
In any right-angled triangle, $a^2 + b^2 = c^2$.
We have $a^2 \equiv b^2 \equiv 1 \pmod 4$, since a and b are odd.
so $c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod 4$,
contradicting the fact that $c^2 \equiv 0$ or $1 \pmod 4$*.

Hence at least one of a and b is even, and $A = (ab)/2$ is always integral.

*If an integer n is even, its square is a multiple of 4 and if it is odd, its square when divided by 4 leaves remainder 1. This fact can be stated as $n^2 \equiv 0$ or $1 \pmod 4$.

So all Pythagorean triangles have integral areas. That is, all Pythagorean triangles are right-angled Heronian triangles and vice versa.

Now, a Pythagorean triple (a, b, c) is said to be primitive if a, b, c have no common factor (written $gcd(a, b, c) = 1$). It is clear that every Pythagorean triple takes the form (ka, kb, kc), where (a, b, c) is a primitive Pythagorean triple and k is some positive integer. It is well known (probably already known to Pythagoras) that all primitive Pythagorean triples (a, b, c) can be obtained in the following way (refer for example Ore, Oystein, "Invitation to Number Theory", New York: Random House. 1967, pp. 49-52):

$a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$, where m, n are positive integers not both odd, $m > n$, and $\gcd(m, n) = 1$.

For example, let $m = 4$ and $n = 1$. then $a = 15$, $b = 8$ and $c = 17$ and (15, 8, 17) is a Pythagorean triple.

We thus know how to generate all right-angled Heronian triangles!

Case 3 Isosceles triangles

> Theorem ***All isosceles Heronian triangles can be divided into two congruent Heronian right-angled triangles.***

By taking any two congruent Pythagorean triangles and putting them side by side in two different ways, we obtain two different isosceles Heronian triangles.

As illustrated in Fig. 1, the Pythagorean triangles with sides (8, 15, 17) can be used to produce two isosceles Heronian triangles with sides (16, 17, 17) and (30, 17, 17) respectively.

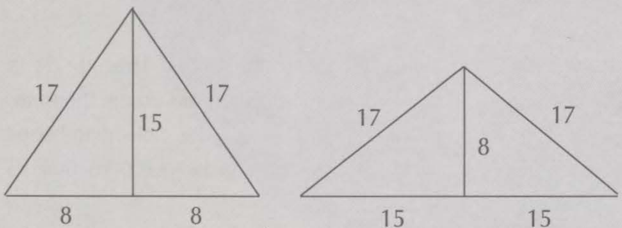


Fig. 1

Conversely, is it true that every isosceles Heronian triangle can be divided into two congruent Pythagorean triangles? We would like to prove that it can indeed be done!

Proof:

Let ABC be an isosceles Heronian triangle with base BC. Let M be the midpoint of BC (see Fig. 2).

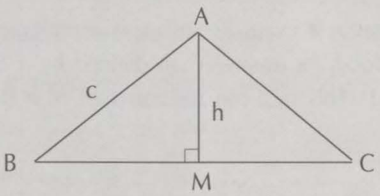


Fig. 2

Assume that a is odd. Let $a = 2n + 1$. Then $BM = n + 1/2$, and

$$h = \sqrt{c^2 - (n + 1/2)^2} = \sqrt{(4c^2 - 4n^2 - 4n - 1)/4}.$$

Let A be the area of ΔABC.

$$\begin{aligned} \text{Then } A &= (2n + 1)/2 \sqrt{(4c^2 - 4n^2 - 4n - 1)/4}, \\ 2A &= (2n + 1) \sqrt{(4c^2 - 4n^2 - 4n - 1)/4}, \\ 4A^2 &= (4n^2 + 4n + 1)(4c^2 - 4n^2 - 4n - 1)/4. \\ \text{So } 16A^2 &= (4n^2 + 4n + 1)(4c^2 - 4n^2 - 4n - 1). \end{aligned}$$

Now l.h.s. is an even number, while r.h.s. is a product of two odd numbers. This is not possible! Hence a must be even. Let $a = 2n$. Then $BM = n$, and

$$h = \sqrt{c^2 - n^2}, \quad A = (2n)/2 \sqrt{c^2 - n^2}, \text{ so that}$$
$$A/n = \sqrt{c^2 - n^2}.$$

Note that $(c^2 - n^2)$ is an integer and A/n is a rational number. Therefore $(c^2 - n^2)$ is a perfect square (if not, A/n would be irrational). Hence h is an integer and (n, h, c) is a Pythagorean triple.

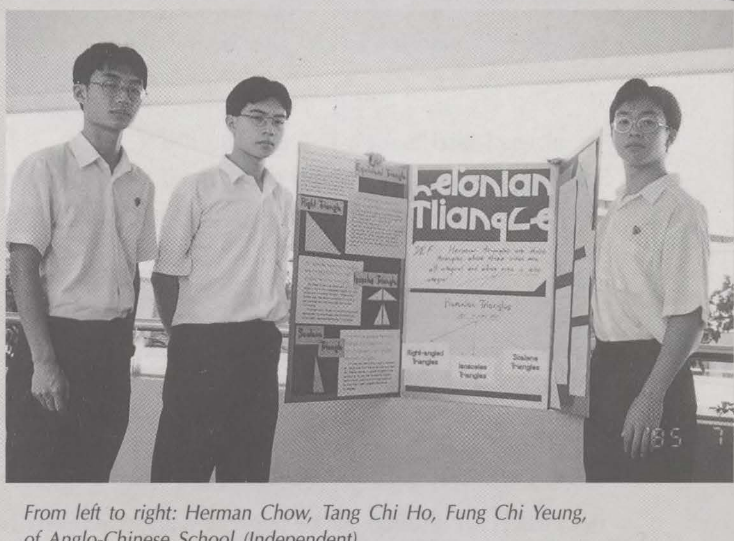
Case 4 Scalene triangles

We know how to generate a lot of scalene Heronian triangles. This is done by simply adjoining two Pythagorean triangles which have exactly one of the legs equal. However a scalene Heronian triangle may not be made up of two Pythagorean triangles. For example the Pythagorean triangle (3, 4, 5) is scalene and it is definitely not made up of two Pythagorean triangles, as (3, 4, 5) is the "smallest" Pythagorean triple. How can we classify the scalene Heronian triangles? The solution seems to be beyond our reach, so it remains an open question in our project.

Summary & Conclusion

In the different cases we examined, the case for the equilateral triangles is easily eliminated. We have ascertained that all right-angled Heronian triangles are the Pythagorean triangles and all isosceles Heronian triangles can be broken up into two congruent Pythagorean triangles. It is easy to form scalene Heronian triangles by joining two Pythagorean triangles with one common leg equal in length. However we were not able to classify the scalene Heronian triangles.

During the process of investigation, several interesting questions come to our mind. For example, is there a Heronian triangle with consecutive integral sides?* More interestingly, is it possible for a tetrahedron to have all six sides, all four surface areas and the volume to be integral? M^2



From left to right: Herman Chow, Tang Chi Ho, Fung Chi Yeung, of Anglo-Chinese School (Independent)

**Editor's note: Yes! The triple (3, 4, 5) is the smallest example. Read the inset for the complete determination of Heronian triangles whose sides are consecutive integers.*

Heronian triples (n - 1, n, n + 1)

by Chua Seng Kiat

We apply Heron's formula. We have

$$s = ((n - 1) + n + (n + 1))/2 = 3n/2 \text{ and}$$

we look for integers n such that

$$\begin{aligned} A &= \sqrt{(3n/2)(3n/2 - (n - 1))(3n/2 - n)(3n/2 - (n + 1))} \\ &= \sqrt{(3n/2)((n + 2)/2)(n/2)((n - 2)/2)} \\ &= (n/4)\sqrt{3(n + 2)(n - 2)} \end{aligned}$$

is an integer.

It is easy to see that n must be even. Set $n = 2m$. Then $A = m\sqrt{3(m + 1)(m - 1)}$. Now set $3(m + 1)(m - 1) = t^2$. t^2 is a multiple of 3, so we can write $t = 3k$ for some integer k. We get $(m + 1)(m - 1) = 3k^2$, so $m^2 - 3k^2 = 1$. Such an equation is known as a Pell's equation. The smallest positive integral solution (m_1, k_1) is (2, 1), which gives the Heronian triple (3, 4, 5). All solutions (m_i, k_i) can be found by the following relation (cf. Niven, Zuckerman & Montgomery, An introduction to the theory of numbers, 5th edition, John Wiley & Son, 1991. Theorem 7.26, p.354):

$$m_i + k_i\sqrt{3} = (2 + \sqrt{3})^i, \quad i = 1, 2, \ldots$$

Examples

i	m_i	k_i	$(n - 1, n, n + 1)$
1	2	1	(3, 4, 5)
2	7	4	(13, 14, 15)
3	26	15	(51, 52, 53)
4	97	56	(193, 194, 195)
5	362	209	(723, 724, 725)
6	1351	780	(2701, 2702, 2703)
7	5042	2911	(10083, 10084, 10085)
8	18817	10864	(37633, 37634, 37635)
9	70226	40545	(140451, 140452, 140453)
10	262087	151316	(524173, 524174, 524175)